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Quantum mechanics on p-adic numbers

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Abstract
A quantum system with positions in $\mathbb{Z}_p$ and momenta in $\mathbb{Q}_p/\mathbb{Z}_p$ is studied. Displacement operators in the phase space of this system and the corresponding Heisenberg–Weyl group $HW[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ are studied. It is shown that such a system can be constructed from a semi-infinite chain of spins which are coupled in a particular way.

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1. Introduction

Finite quantum systems where the position and momentum take values in the ring $\mathbb{Z}(d)$ (the integers modulo d) were studied originally by Weyl [1] and Schwinger [2] and later by many authors (e.g., [3–7]). The mathematical work in [8, 9] is also related to this. We have reviewed this work in [10].

We use a notation which shows clearly the $G \times \widetilde{G}$ position-momentum phase space, where G is the additive Abelian group of positions and $\widetilde{G}$ is the Pontyagin dual group of momenta. In this notation we call these systems $S[\mathbb{Z}(d) \times \mathbb{Z}(d)]$. We also call their d-dimensional Hilbert space $\mathcal{H}[\mathbb{Z}(d) \times \mathbb{Z}(d)]$ and the corresponding Heisenberg–Weyl group of displacements $HW[\mathbb{Z}(d) \times \mathbb{Z}(d)]$.

When d is equal to a prime number p, these systems have stronger properties (e.g., there are well-defined symplectic transformations, the number of mutually unbiased bases reaches its maximum value $d+1$, etc). This is intimately related to the fact that for prime p the position and momentum take values in the field $\mathbb{Z}(p)$, and the existence of inverses leads to stronger properties.

A quantum system where the position and momentum take values in the Galois field $GF(p^\ell)$ has also been studied. In our notation this is the $S[GF(p^\ell) \times GF(p^\ell)]$ system and its Hilbert space $\mathcal{H}[GF(p^\ell) \times GF(p^\ell)]$ is p^ℓ-dimensional. The Heisenberg–Weyl $HW[GF(p^\ell) \times GF(p^\ell)]$ group of displacements, the symplectic $Sp(2, GF(p^\ell))$ group and Frobenius transformations in this system have been studied in [11, 12] (and reviewed in [13]). Mutually unbiased bases in this system have been studied extensively in recent years (e.g., [14–17]).

In a different line of research, quantum mechanics and quantum field theory in the field $\mathbb{Q}_p$ of p-adic numbers [18] have been studied extensively in the literature [19–22]. In our notation this is the study of $S[\mathbb{Q}_p \times \mathbb{Q}_p]$ systems. The mathematical background for quantum mechanics on locally compact fields (e.g., Fourier transforms and special functions on locally compact fields) is dicussed in [23, 24]. Work on wavelets on locally compact fields is also related [25, 27].

In this paper we consider a $S[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ system where the position takes values in the ring $\mathbb{Z}_p$ of p-adic integers and consequently the momentum takes values in $\mathbb{Q}_p/\mathbb{Z}_p$. Displacement operators in this system and their properties, and the corresponding Heisenberg–Weyl group $HW[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ are studied.

The literature on p-adic physics has studied both cases of complex wavefunctions (of p-adic variables) and also p-adic valued wavefunctions (of p-adic variables). We use complex wavefunctions of p-adic variables.

In this paper we consider the case of a fixed prime number p. If we consider all prime numbers we naturally go into quantum mechanics on adeles [27].

From a physical point of view, most of the literature on p-adic physics assumes that spacetime has a p-adic structure at the Planck scale. Here we are interested in a very different physical application, namely the 'quantum-engineering' of a $S[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ system from a semi-infinite chain of p-dimensional quantum systems, e.g., spins with $j = (p-1)/2$. We show that if these spins are coupled in a certain way (which is described by a certain class of Hamiltonians) then this system acquires a p-adic structure. This means that there is a one-to-one linear map between the quantum states in the chain of spins and the quantum states of a $S[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ system, which is preserved as the two systems evolve in time. Such 'quantum devices with p-adic structure' might be useful for technology. From a theoretical point of view, there is merit in understanding how a semi-infinite chain of spins gets p-adic structure. $\mathbb{Z}_p$ is a profinite group [28], i.e., a Hausdorff compact totally disconnected topological group and it is interesting to see how the semi-infinite chain of spins acquires this structure.

In section 2 we discuss aspects of p-adic numbers which are used later. In section 3 we consider a quantum system with positions in $\mathbb{Z}_p$ and momenta in $\mathbb{Q}_p/\mathbb{Z}_p$. We study displacement operators in the phase space of this system, the corresponding Heisenberg–Weyl group $HW[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$, and coherent states.

In section 4 we start with a very brief review of $S[\mathbb{Z}(p) \times \mathbb{Z}(p)]$ quantum systems with p-dimensional Hilbert space, in order to establish the notation. We note that some of the results for finite quantum systems are valid only in the case of an odd dimension, and the case of even dimension requires special consideration (e.g., [10]). For this reason, below p is an odd prime number. We then introduce a family of states and operators which we call Prüfer states and operators because they are related to the Prüfer (or quasi-cyclic) p-group. Using them, we show in section 5 that a chain of $\ell S[\mathbb{Z}(p) \times \mathbb{Z}(p)]$ systems (e.g., ℓ spins with $j = (p-1)/2$) which are coupled in a certain way is 'equivalent' to a $S[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ system. The term 'equivalent' means that there is a one-to-one linear map between the Hilbert spaces of the two systems which is preserved as the two systems evolve in time. The coupling between the spins introduces a hierarchy among them, which gives the system the p-adic structure.

We conclude in section 6 with a discussion of our results.

2. The field $\mathbb{Q}_p$ of p-adic numbers

The field $\mathbb{Q}_p$ is a locally compact field of characteristic zero. Topologically it is a totally disconnected Hausdorff topological space. An element in $\mathbb{Q}_p$ can be written as

$$\alpha = \sum_{\nu=\mathrm{ord}(\alpha)}^{\infty} \overline{\alpha}_\nu p^\nu; \qquad 0 \leqslant \overline{\alpha}_\nu \leqslant p-1. \tag{1}$$

Addition and multiplication of two elements is the usual addition and multiplication for series together with the 'carrying' of digits, so that $0 \leqslant \overline{\alpha}_\nu \leqslant p-1$. The metric is non-Archimedian and the absolute value of α is given by

$$|\alpha| = p^{-\mathrm{ord}(\alpha)}. \tag{2}$$

We use the notation $\mathbb{Z}_p$ for the ring of integers in $\mathbb{Q}_p$ (i.e.,$|\alpha| \leqslant 1$). We also use the notation $p^\mu\mathbb{Z}_p$ for the set of all α such that $|\alpha| \leqslant p^{-\mu}$ and the notation $\beta + p^\mu\mathbb{Z}_p$ for the set of all $\alpha+\beta$ where $|\alpha| \leqslant p^{-\mu}$.

$\mathbb{Z}_p$ is an additive profinite group [28], and as such it is a Hausdorff compact totally disconnected topological group. The

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset \ldots \tag{3}$$

is a fundamental system of neighbourhoods of 0.

An element $\mathfrak{p}$ in $\mathbb{Q}_p/\mathbb{Z}_p$ is a coset and we represent it with the element which has integer part equal to zero:

$$\mathfrak{p} = \overline{\mathfrak{p}}_{-k}p^{-k} + \overline{\mathfrak{p}}_{-k+1}p^{-k+1} + \cdots + \overline{\mathfrak{p}}_{-1}p^{-1}; \qquad 0 \leqslant \overline{\mathfrak{p}}_i \leqslant p-1; \qquad k = -\mathrm{ord}(\mathfrak{p}). \tag{4}$$

The product $x\mathfrak{p}$ where $x \in \mathbb{Z}_p$ and $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$ is also a coset in $\mathbb{Q}_p/\mathbb{Z}_p$, and we represent it with the element which has integer part equal to zero, as

$$x\mathfrak{p} = \sum_{\nu=\mathrm{ord}(\mathfrak{p})}^{-1} \overline{\alpha}_\nu p^\nu. \tag{5}$$

Here the multiplication $x\mathfrak{p}$ leads to a series $\sum \alpha_\nu p^\nu$ where $\alpha_\nu = \sum_{\mu=\mathrm{ord}(\mathfrak{p})}^{\nu} \overline{\mathfrak{p}}_\mu \overline{x}_{\nu-\mu}$. Since α_ν are not necessarily less than p we perform the 'carrying' operation and we get the series $\sum \overline{\alpha}_\nu p^\nu$ which we truncate at $\nu = -1$ to get the above result.

The Schwartz–Bruhat space of complex functions $f(x)$ (where $x \in \mathbb{Q}_p$) which are locally constant with degree n and have compact support with degree k, consists of functions for which there exist integers k, n such that

$$\begin{aligned} &f(x) = 0 \qquad \text{for} \quad |x| > p^k, \\ &f(x+\alpha) = f(x) \qquad \text{for} \quad |\alpha| \leqslant p^{-n}. \end{aligned} \tag{6}$$

2.1. Integrals over $\mathbb{Z}_p$

In integrals we use the Haar measure, normalized as

$$\int_{\mathbb{Z}_p} \mathrm{d}x = 1. \tag{7}$$

For practical calculations we mention the following formulae [24]:

$$\int_{p^k\mathbb{Z}_p} \mathrm{d}x = p^{-k}; \qquad \int_{|x|=p^k} \mathrm{d}x = p^k - p^{k-1}. \tag{8}$$

The integral of a locally constant function of degree n is given by

$$\int_{\mathbb{Z}_p} f(x)\,\mathrm{d}x = p^{-n} \sum f(\overline{x}_0 + \overline{x}_1 p + \cdots + \overline{x}_{n-1}p^{n-1}). \tag{9}$$

Here we sum over all $\{\overline{x}_0, \ldots, \overline{x}_{n-1}\}$. In the special case that $f(x) = 1$ for all $x \in \mathbb{Z}_p$, equation (9) reduces to equation (7).

2.2. Integrals over $\mathbb{Q}_p/\mathbb{Z}_p$

Let $f(\mathfrak{p})$ where $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$ be a complex function which has compact support with degree k. The integral of this function over $\mathbb{Q}_p/\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} f(\mathfrak{p})\,\mathrm{d}\mathfrak{p} = \sum f(\overline{\mathfrak{p}}_{-k}p^{-k} + \overline{\mathfrak{p}}_{-k+1}p^{-k+1} + \cdots + \overline{\mathfrak{p}}_{-1}p^{-1}), \tag{10}$$

where the summation is over all $\{\overline{\mathfrak{p}}_{-k}, \overline{\mathfrak{p}}_{-k+1}, \ldots, \overline{\mathfrak{p}}_{-1}\}$. The counting measure is used on $\mathbb{Q}_p/\mathbb{Z}_p$ (see the appendix). In the special case that $f(\mathfrak{p}) = 1$ for all $|\mathfrak{p}| = p^k$, and $f(\mathfrak{p}) = 0$ elsewhere, equation (10) reduces to equation (8).

More generally, let $f(\mathfrak{p})$ where $\mathfrak{p} \in \mathbb{Q}_p/p^n\mathbb{Z}_p$ be a complex function which has compact support with degree k. The integral of this function over $\mathbb{Q}_p/p^n\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Q}_p/p^n\mathbb{Z}_p} f(\mathfrak{p})\,\mathrm{d}\mathfrak{p} = p^{-n}\sum f(\overline{\mathfrak{p}}_{-k}p^{-k} + \overline{\mathfrak{p}}_{-k+1}p^{-k+1} + \cdots + \overline{\mathfrak{p}}_{n-1}p^{n-1}). \tag{11}$$

2.3. Delta functions

Delta functions in the present context have been discussed in [24]. The $\delta(x-y)$ where $x, y \in \mathbb{Z}_p$ is such that if $f(x)$ is a complex function with $x \in \mathbb{Z}_p$, then

$$\int_{\mathbb{Z}_p} f(x)\delta(x-y)\,\mathrm{d}x = f(y); \qquad x, y \in \mathbb{Z}_p, \tag{12}$$

$\delta(x-y)$ is zero everywhere apart from the point $x = y$, where it is infinite. Delta functions are not locally constant and therefore they do not belong to the Schwartz-Bruhat space of locally costant functions with compact support. They are generalized functions which belong to a rigged Hilbert space (constructed with a Gel'fand triplet). This is analogous to the harmonic oscillator case and in the present context of p-adic numbers is discussed in [24].

The $\Delta(\mathfrak{p}-\mathfrak{q})$ where $\mathfrak{p}, \mathfrak{q} \in \mathbb{Q}_p/\mathbb{Z}_p$ is similar to a Kronecker delta:

$$\begin{aligned} \Delta(\mathfrak{p}-\mathfrak{q}) &= 1; \qquad \text{if} \quad \mathfrak{p} = \mathfrak{q}, \\ \Delta(\mathfrak{p}-\mathfrak{q}) &= 0; \qquad \text{if} \quad \mathfrak{p} \neq \mathfrak{q}. \end{aligned} \tag{13}$$

If $f(\mathfrak{p})$ is a complex function with $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$, then equation (10) shows that

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} f(\mathfrak{p})\Delta(\mathfrak{p}-\mathfrak{q})\,\mathrm{d}\mathfrak{p} = f(\mathfrak{q}); \qquad \mathfrak{p}, \mathfrak{q} \in \mathbb{Q}_p/\mathbb{Z}_p. \tag{14}$$

2.4. Additive characters

An additive character $\chi(\alpha)$ of $\alpha = \sum \overline{\alpha}_\nu p^\nu$ (where $0 \leqslant \overline{\alpha}_\nu \leqslant p-1$) is given by

$$\begin{aligned} \chi(\alpha) &= \exp\left(\mathrm{i}2\pi \sum_{\nu=\mathrm{ord}(\alpha)}^{-1} \overline{\alpha}_\nu p^\nu\right); \qquad \mathrm{ord}(\alpha) \leqslant -1 \\ \chi(\alpha) &= 1; \qquad \mathrm{ord}(\alpha) \geqslant 0. \end{aligned} \tag{15}$$

It is a locally constant function of degree $n = 0$ and obeys the relation

$$\chi(\alpha)\chi(\beta) = \chi(\alpha+\beta). \tag{16}$$

The additive character $\chi(\alpha\beta)$ of the product of $\alpha = \sum \overline{\alpha}_\nu p^\nu$ and $\beta = \sum \overline{\beta}_\mu p^\mu$ is given by

$$\chi(\alpha\beta) = \exp\left(\mathrm{i}2\pi \sum_{\nu,\mu} \overline{\alpha}_\nu \overline{\beta}_\mu p^{\nu+\mu}\right); \qquad \nu + \mu \leqslant -1; \qquad 0 \leqslant \overline{\alpha}_\nu, \quad \overline{\beta}_\mu \leqslant p - 1. \tag{17}$$

We can show that

$$\int_{\mathbb{Z}_p} \chi(x\mathfrak{p})\,\mathrm{d}x = \Delta(\mathfrak{p}); \qquad \int_{\mathbb{Q}_p/\mathbb{Z}_p} \chi(x\mathfrak{p})\,\mathrm{d}\mathfrak{p} = \delta(x), \tag{18}$$

where $x \in \mathbb{Z}_p$ and $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$. In order to prove this we express the first integral as

$$\begin{aligned}\int_{\mathbb{Z}_p} \chi(x\mathfrak{p})\,\mathrm{d}x &= \lim_{\ell\to\infty} p^{-\ell} \sum_{\{\overline{x}_\nu\}} \exp\left(\mathrm{i}2\pi \sum_{\nu,\kappa} \frac{\overline{x}_\nu \overline{\mathfrak{p}}_{-\nu-\kappa}}{p^\kappa}\right) \\ &= \lim_{\ell\to\infty} p^{-\ell} \prod_{\nu,\kappa} \left[\sum_{\{\overline{x}_\nu\}} \exp\left(\mathrm{i}2\pi \frac{\overline{x}_\nu \overline{\mathfrak{p}}_{-\nu-\kappa}}{p^\kappa},\right)\right],\end{aligned} \tag{19}$$

where $1 \leqslant k \leqslant \ell - 1$ and $0 \leqslant \overline{x}_\nu, \overline{\mathfrak{p}}_{-\nu} \leqslant p - 1$. Then we use the relation

$$\frac{1}{p} \sum_{\overline{x}_\nu \in \mathbb{Z}(p)} \exp\left(\mathrm{i}2\pi \frac{\overline{x}_\nu \overline{\mathfrak{p}}_{-\nu-1}}{p}\right) = \delta(\overline{\mathfrak{p}}_{-\nu-1}, 0), \tag{20}$$

where δ is Kronecker's delta. This shows that the result is non-zero only if all $\overline{\mathfrak{p}}_{-\nu-1} = 0$, i.e., $\mathfrak{p} = 0$. When $\mathfrak{p} = 0$, $\int_{\mathbb{Z}_p} \chi(x\mathfrak{p})\,\mathrm{d}x = \int_{\mathbb{Z}_p} \mathrm{d}x = 1$. In a similar way we prove the second relation in equation (18).

2.5. The Prüfer p-group

The multiplicative cyclic group $C(p^\ell)$ is

$$C(p^\ell) = \{\omega_\ell(\alpha_\ell) | \alpha_\ell \in \mathbb{Z}(p^\ell)\}, \tag{21}$$

where

$$\omega_\ell(\alpha_\ell) = \exp\left(\frac{\mathrm{i}2\pi\alpha_\ell}{p^\ell}\right); \qquad \alpha_\ell \in \mathbb{Z}(p^\ell) \tag{22}$$

are roots of unity. $C(p^\ell)$ is the Pontryagin dual of $\mathbb{Z}(p^\ell)$ and is isomorphic to $\mathbb{Z}(p^\ell)$, i.e., $C(p^\ell) \cong \mathbb{Z}(p^\ell)$.

The Prüfer p-group (or quasi-cyclic group) $C(p^\infty)$ contains all p^ℓ th roots of unity (for all $\ell \in \mathbb{Z}^+$):

$$C(p^\infty) = \{\omega_\ell(\alpha_\ell) | \alpha_\ell \in \mathbb{Z}(p^\ell), \ell \in \mathbb{Z}^+\}. \tag{23}$$

The Prüfer p-group is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ (the multiplication in $C(p^\infty)$ corresponds to addition in $\mathbb{Q}_p/\mathbb{Z}_p$). The correspondence between the elements of the two groups is given by

$$\mathbb{Q}_p/\mathbb{Z}_p \ni \alpha \quad \leftrightarrow \quad \chi(\alpha) = \omega_\nu(p^\nu\alpha) \in C(p^\infty); \qquad \nu = -\mathrm{ord}(\alpha) \tag{24}$$

which we can also rewrite as

$$\alpha = \overline{\alpha}_{-\nu} p^{-\nu} + \cdots + \overline{\alpha}_{-1} p^{-1} \leftrightarrow \chi(\alpha) = \omega_\nu(\overline{\alpha}_{-\nu}) \ldots \omega_1(\overline{\alpha}_{-1}). \tag{25}$$

3. Quantum systems with positions in $\mathbb{Z}_p$ and momenta in $\mathbb{Q}_p/\mathbb{Z}_p$

We consider quantum systems with positions $x \in \mathbb{Z}_p$. The Pontryagin dual of $\mathbb{Z}_p$ is $\mathbb{Q}_p/\mathbb{Z}_p$ and therefore the momentum $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$. We denote as $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ the Hilbert space of this system. It comprises complex functions $f(x)$ where $x \in \mathbb{Z}_p$, which are locally constant and have compact support. The scalar product is given by

$$\langle f|g\rangle = \int_{\mathbb{Z}_p} [f(x)]^* g(x)\,\mathrm{d}x. \tag{26}$$

For convinience we normalize these functions to one($\langle f|f\rangle = 1$).

The Fourier transform of these functions is given by

$$\tilde{f}(\mathfrak{p}) = \int_{\mathbb{Z}_p} \mathrm{d}x\, \chi(-x\mathfrak{p}) f(x); \qquad \mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p. \tag{27}$$

The inverse Fourier transform is given by

$$f(x) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\, \chi(x\mathfrak{p}) \tilde{f}(\mathfrak{p}); \qquad x \in \mathbb{Z}_p. \tag{28}$$

3.1. Position and momentum states

We consider an orthonormal basis $|\mathsf{X}; x\rangle$ which we call position states:

$$\int_{\mathbb{Z}_p} \mathrm{d}x |\mathsf{X}; x\rangle\langle \mathsf{X}; x| = \mathbf{1}, \qquad \langle \mathsf{X}; x|\mathsf{X}; y\rangle = \delta(x-y). \tag{29}$$

The X in the notation is not a variable, but it simply indicates position states. The wavefunctions of the position states are delta functions and we have already mentioned that they do not belong to the Hilbert space $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$. They are rigorously introduced in the corresponding rigged Hilbert space formalism, discussed in the context of p-adic numbers in [24]. The same is true about the corresponding position operator introduced below.

Through Fourier transform we introduce another basis, the momentum states

$$|\mathsf{P}; \mathfrak{p}\rangle = \int_{\mathbb{Z}_p} \mathrm{d}x\, \chi(x\mathfrak{p}) |\mathsf{X}; x\rangle. \tag{30}$$

The P in the notation is not a variable, but it simply indicates momentum states. They also form an orthonormal basis

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p} |\mathsf{P}; \mathfrak{p}\rangle\langle \mathsf{P}; \mathfrak{p}| = \mathbf{1}; \qquad \langle \mathsf{P}; \mathfrak{p}|\mathsf{P}; \mathfrak{q}\rangle = \Delta(\mathfrak{p}-\mathfrak{q}). \tag{31}$$

An arbitrary state $|f\rangle$ in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$, normalized so that $\langle f|f\rangle = 1$, can be expanded in these two bases as

$$|f\rangle = \int_{\mathbb{Z}_p} \mathrm{d}x\, f(x) |\mathsf{X}; x\rangle = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\, \tilde{f}(\mathfrak{p}) |P; \mathfrak{p}\rangle. \tag{32}$$

We also introduce the following operators:

$$U_n(\mathfrak{a}) = \int_{\mathbb{Z}_p} \chi(\mathfrak{a}x^n) |\mathsf{X}; x\rangle\langle \mathsf{X}; x|\,\mathrm{d}x, \qquad V_n(b) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \chi(b\mathfrak{p}^n) |\mathsf{P}; \mathfrak{p}\rangle\langle \mathsf{P}; \mathfrak{p}|\,\mathrm{d}\mathfrak{p}. \tag{33}$$

Formally these operators can be written as

$$U_n(\mathfrak{a}) = \chi(\mathfrak{a}\widehat{\mathsf{X}}^n), \qquad V_n(b) = \chi(b\widehat{\mathsf{P}}^n), \tag{34}$$

where $\widehat{\mathsf{X}}$, $\widehat{\mathsf{P}}$ are the position and momentum operators

$$\widehat{\mathsf{X}} = \int_{\mathbb{Z}_p} x|\mathsf{X}; x\rangle\langle\mathsf{X}; x|\,\mathrm{d}x; \qquad \widehat{\mathsf{P}} = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathfrak{p}|\mathsf{P}; \mathfrak{p}\rangle\langle\mathsf{P}; \mathfrak{p}|\,\mathrm{d}\mathfrak{p}. \tag{35}$$

It has been pointed out by various authors that the multiplication $xf(x)$ is not defined because x is a p-adic number and $f(x)$ is a complex function. The $\widehat{\mathsf{X}}$, $\widehat{\mathsf{P}}$ are meaningful only through equations (33) and (34) which use characters of x times a complex function. It is clear that the use of $\widehat{\mathsf{X}}$, $\widehat{\mathsf{P}}$ is limited. We introduce them for two reasons. The first is that below we consider finite-dimensional subspaces of $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ where the positions and momentum operators are defined. The second reason is that we can use them as a guide in order to introduce Hamiltonians $H(\widehat{\mathsf{X}}, \widehat{\mathsf{P}})$ analogous to those in a harmonic oscillator. One example is

$$\begin{aligned} \exp(\mathrm{i}H_1) &= U_2(\mathfrak{a})V_2(b) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \mathrm{d}x\,\mathrm{d}y\, f(x,y)|\mathsf{X}; x\rangle\langle\mathsf{X}; y| \\ f(x,y) &= \chi(\mathfrak{a}x^2)\int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\,\chi(b\mathfrak{p}^2 + x\mathfrak{p} - y\mathfrak{p}). \end{aligned} \tag{36}$$

There is no simple relation between this and $\exp(\mathrm{i}H_2) = V_2(b)U_2(\mathfrak{a})$ but they are both p-adic analogues of the exponential of the harmonic oscillator Hamiltonian. Another example is $\exp(\mathrm{i}H_3) = V_2(b)U_2(\mathfrak{a}_1)U_4(\mathfrak{a}_2)$ which is the p-adic analogue of the exponential of the Hamiltonian of the quartic oscillator.

3.2. The subspace $\mathcal{H}_\ell$ of locally constant functions with degree ℓ

If $f(x)$ has compact support with degree k and is locally constant with degree n, then $\tilde{f}(\mathfrak{p})$ has compact support with degree n and is locally constant with degree k. To prove this we assume that

$$f(x+\alpha) = f(x) \qquad \text{for} \quad |\alpha| \leqslant p^{-n} \tag{37}$$

and rewrite this in terms of $\tilde{f}(\mathfrak{p})$ as

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\,\chi(x\mathfrak{p})\tilde{f}(\mathfrak{p})[1 - \chi(\alpha\mathfrak{p})] = 0 \qquad \text{for} \quad |\alpha| \leqslant p^{-n}. \tag{38}$$

This shows that $\tilde{f}(\mathfrak{p}) = 0$ for $|\alpha| \leqslant p^{-n}$ and $\mathfrak{p} > p^n$ (so that $\chi(\alpha\mathfrak{p}) \neq 1$). Therefore the function $\tilde{f}(\mathfrak{p})$ has compact support with degree n. In a similar way we prove that if $f(x)$ has compact support with degree k then $\tilde{f}(\mathfrak{p})$ is locally constant with degree k.

We consider the p^ℓ-dimensional space $\mathcal{H}_\ell$ which consists of functions $f(x)$ where $x \in \mathbb{Z}_p/p^\ell\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)$. These functions have compact support with degree 0 and are locally constant with degree ℓ. Their Fourier transforms $\tilde{f}(\mathfrak{p})$ where $\mathfrak{p} \in p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)$ have compact support with degree ℓ and are locally constant with degree 0,

$$\begin{aligned} \mathcal{H}_\ell &= \{f(x)|x \in \mathbb{Z}_p/p^\ell\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)\} \\ &= \{\tilde{f}(\mathfrak{p})|\mathfrak{p} \in p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)\}. \end{aligned} \tag{39}$$

The space $\mathcal{H}_\ell$ is a subspace of $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ and $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \ldots$.

Let Π_ℓ be the projection operator from $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ to $\mathcal{H}_\ell$. Position and momentum operators acting on $\mathcal{H}_\ell$ are given by

$$\begin{aligned} \widehat{\mathsf{X}}_\ell &= \int_{\mathbb{Z}_p/p^\ell\mathbb{Z}_p} x|\mathsf{X}; x\rangle\langle\mathsf{X}; x|\,\mathrm{d}x = \Pi_\ell\widehat{\mathsf{X}}\Pi_\ell \\ \widehat{\mathsf{P}}_\ell &= \int_{p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p} \mathfrak{p}|\mathsf{P}; \mathfrak{p}\rangle\langle\mathsf{P}; \mathfrak{p}|\,\mathrm{d}\mathfrak{p} = \Pi_\ell\widehat{\mathsf{P}}\Pi_\ell. \end{aligned} \tag{40}$$

Here the operators $\widehat{\mathsf{X}}_\ell$ and $\widehat{\mathsf{P}}_\ell$ are defined. They are $p^\ell \times p^\ell$ matrices with elements in $\mathbb{Z}_p/p^\ell\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)$ and $p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^\ell)$. We note however that here also the use of the position and momentum operators is limited, because in finite systems the Heisenberg–Weyl group is discrete and the commutatator $[\widehat{\mathsf{X}}_\ell, \widehat{\mathsf{P}}_\ell]$ is not useful [10, 13]. It is the exponential of these operators (the analogues of equation (34)) that are useful).

We have already introduced Fourier transform in equations (27) and (28). For later use, we also introduce the Fourier operator which acts on various states in $\mathcal{H}_\ell$ to produce their Fourier transforms:

$$\begin{aligned}\mathsf{F}_\ell &= p^{\ell/2}\int_{\mathbb{Z}_p/p^\ell\mathbb{Z}_p} \mathrm{d}x \int_{\mathbb{Z}_p/p^\ell\mathbb{Z}_p} \mathrm{d}y\, \chi\left(\frac{xy}{p^\ell}\right)|X; y\rangle\langle X; x| \\ &= p^{-\ell/2}\int_{p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p} \int_{p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}'\, \chi(\mathfrak{p}\mathfrak{p}'p^\ell)|P; \mathfrak{p}\rangle\langle P; \mathfrak{p}'| \\ &= p^{\ell/2}\int_{\mathbb{Z}_p/p^\ell\mathbb{Z}_p} \mathrm{d}x |P; p^{-\ell}x\rangle\langle X; x|.\end{aligned} \tag{41}$$

It is easily seen that $\mathsf{F}_\ell^4 = \mathbf{1}_\ell$ where $\mathbf{1}_\ell$ is the unity operator within $\mathcal{H}_\ell$.

We can prove that

$$\mathsf{F}_\ell\mathsf{X}_\ell\mathsf{F}_\ell^\dagger = \mathsf{P}_\ell. \tag{42}$$

3.3. The Heisenberg–Weyl group $HW[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$

The displacement operators $\mathsf{Z}(\mathfrak{a})$ and $\mathsf{X}(b)$ are defined as follows:

$$\begin{aligned}\mathsf{Z}(\mathfrak{a}) &= \int_{\mathbb{Z}_p} \mathrm{d}x\, \chi(\mathfrak{a}x)|\mathsf{X}; x\rangle\langle \mathsf{X}; x| = \chi(\mathfrak{a}\widehat{\mathsf{X}}), \\ \mathsf{X}(b) &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\, \chi(-b\mathfrak{p})|\mathsf{P}; \mathfrak{p}\rangle\langle \mathsf{P}; \mathfrak{p}| = \chi(-b\widehat{\mathsf{P}}).\end{aligned} \tag{43}$$

More general displacement operators are defined as

$$\mathsf{D}(\mathfrak{a}, b, \mathfrak{c}) = \mathsf{Z}(\mathfrak{a})\mathsf{X}(b)\chi\left(\mathfrak{c} - \tfrac{1}{2}\mathfrak{a}b\right); \qquad [\mathsf{D}(\mathfrak{a}, b, \mathfrak{c})]^\dagger = \mathsf{D}(-\mathfrak{a}, -b, -\mathfrak{c}), \tag{44}$$

where $\mathfrak{a}, \mathfrak{c} \in \mathbb{Q}_p/\mathbb{Z}_p$ and $b \in \mathbb{Z}_p$.

Using equation (18) we prove that

$$\mathsf{Z}(\mathfrak{a})|\mathsf{P}; \mathfrak{p}\rangle = |\mathsf{P}; \mathfrak{p} + \mathfrak{a}\rangle; \qquad \mathsf{X}(b)|\mathsf{X}; x\rangle = |\mathsf{X}; x + b\rangle. \tag{45}$$

We use these relations to prove that

$$\mathsf{X}(b)\mathsf{Z}(\mathfrak{a}) = \mathsf{Z}(\mathfrak{a})\mathsf{X}(b)\chi(-\mathfrak{a}b) \tag{46}$$

and more generally that

$$\mathsf{D}(\mathfrak{a}, b, \mathfrak{c})\mathsf{D}(\mathfrak{a}', b', \mathfrak{c}') = \mathsf{D}[\mathfrak{a} + \mathfrak{a}', b + b', \mathfrak{c} + \mathfrak{c}' + 2^{-1}(\mathfrak{a}b' - \mathfrak{a}'b)]. \tag{47}$$

Therefore the $\mathsf{D}(\mathfrak{a}, b, \mathfrak{c})$ form a representation of the Heisenberg–Weyl group, which we denote as $HW[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$.

We consider the following subgroups of $HW[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$

$$\begin{aligned}\mathsf{G} &= \{\mathsf{Z}(\mathfrak{a})|\mathfrak{a} \in \mathbb{Q}_p/\mathbb{Z}_p\} \cong \mathbb{Q}_p/\mathbb{Z}_p, \\ \mathsf{K} &= \{\mathsf{X}(b)|b \in \mathbb{Z}_p\} \cong \mathbb{Z}_p.\end{aligned} \tag{48}$$

G is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ and therefore it is discrete. K is isomorphic to $\mathbb{Z}_p$ and therefore it is a profinite group. We define its subgroups as

$$\mathsf{K}_n = \{\mathsf{X}(b)|b \in p^n\mathbb{Z}_p\} \cong p^n\mathbb{Z}_p. \tag{49}$$

The

$$\mathsf{K} \supset \mathsf{K}_1 \supset \mathsf{K}_2 \supset \ldots \tag{50}$$

is a fundamental system of neighbourhoods of **1**.

3.4. Coherent states

Let Θ be a trace class operator acting on $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$. We show that

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{a} \int_{\mathbb{Z}_p} \mathrm{d}b\, D(\mathfrak{a}, b, 0)\Theta[D(\mathfrak{a}, b, 0)]^\dagger = [\mathrm{tr}\,\Theta]\mathbf{1}. \tag{51}$$

In order to prove this we consider the matrix elements of both sides with respect to $\langle X; x_1|$ and $|X; x_2\rangle$, and substitute the matrix elements

$$\langle X; x_1|D(\mathfrak{a}, b, 0)|X; x_2\rangle = \chi\left(\tfrac{1}{2}\mathfrak{a}b + \mathfrak{a}x_2\right)\delta(x_1 - x_2 - b) \tag{52}$$

and perform the integration.

Let $|s\rangle$ be an arbitrary (normalized) 'fiducial' vector in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$. We define coherent states as

$$|\mathfrak{a}, b; s\rangle \equiv D(\mathfrak{a}, b, 0)|s\rangle; \qquad b \in \mathbb{Z}_p; \qquad \mathfrak{a} \in (\mathbb{Q}_p/\mathbb{Z}_p). \tag{53}$$

Then equation (51) with $\Theta = |s\rangle\langle s|$ reduces to the resolution of the identity in terms of coherent states:

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{a} \int_{\mathbb{Z}_p} \mathrm{d}b|\mathfrak{a}, b; s\rangle\langle\mathfrak{a}, b; s| = \mathbf{1}. \tag{54}$$

Using coherent states we can represent a state $|f\rangle$ in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ with the complex function

$$f_B(\mathfrak{a}, b; s) = \langle\mathfrak{a}, b; s|f\rangle; \qquad b \in \mathbb{Z}_p; \qquad \mathfrak{a} \in (\mathbb{Q}_p/\mathbb{Z}_p). \tag{55}$$

The index 'B' in the notation stands for Bargmann, because this is reminiscent of the Bargmann representation in the harmonic oscillator formalism (although there is no analyticity here). Then equation (54) shows that the scalar product of two states is given by

$$\langle f|g\rangle = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{a} \int_{\mathbb{Z}_p} \mathrm{d}b[f_B(\mathfrak{a}, b; s)]^* g_B(\mathfrak{a}, b; s). \tag{56}$$

Let $\sigma(x; \mathfrak{a}, b)$ be the wavefunction of the coherent state

$$\sigma(x; \mathfrak{a}, b; s) = \langle x|\mathfrak{a}, b; s\rangle. \tag{57}$$

The wavefunctions $f_B(\mathfrak{a}, b; s)$ and $f(x)$ of a state $|f\rangle$ are related to each other as follows:

$$\begin{aligned} f_B(\mathfrak{a}, b; s) &= \int_{\mathbb{Z}_p} \mathrm{d}x[\sigma(x; \mathfrak{a}, b; s)]^* f(x), \\ f(x) &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{a} \int_{\mathbb{Z}_p} \mathrm{d}b\, \sigma(x; \mathfrak{a}, b; s) f_B(\mathfrak{a}, b; s). \end{aligned} \tag{58}$$

A trace-class operator Θ acting on $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ is represented with the function

$$\Theta_B(\mathfrak{a}, b; \mathfrak{a}', b'; s) = \langle\mathfrak{a}, b; s|\Theta|\mathfrak{a}', b'; s\rangle. \tag{59}$$

It acts on a state $|f\rangle$ as follows:

$$\Theta|f\rangle \to \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{a}' \int_{\mathbb{Z}_p} \mathrm{d}b'\, \Theta_B(\mathfrak{a}, b; \mathfrak{a}', b'; s) f_B(\mathfrak{a}', b'; s). \tag{60}$$

3.5. Example

We consider p-adic numbers with $p = 3$ and the state $|s\rangle$ with wavefunction $s(x) = \langle x|s\rangle$ (where $x \in \mathbb{Z}_p$) given by

$$\begin{aligned} s(x) &= 1.5^{1/2}; && \text{if} \quad x \in p\mathbb{Z}_p \\ s(x) &= 0.6^{1/2}; && \text{if} \quad x \in 1 + p\mathbb{Z}_p \\ s(x) &= 0.9^{1/2}; && \text{if} \quad x \in 2 + p\mathbb{Z}_p. \end{aligned} \tag{61}$$

This is a locally constant function with degree $n = 1$ and has compact support with degree $k = 0$. It is normalized so that $\langle s|s\rangle = 1$.

Its Fourier transform is

$$\tilde{s}(\mathfrak{p}) = 1.5^{1/2} \int_{p\mathbb{Z}_p} \mathrm{d}x\, \chi(-x\mathfrak{p}) + 0.9^{1/2} \int_{1+p\mathbb{Z}_p} \mathrm{d}x\, \chi(-x\mathfrak{p}) + 0.6^{1/2} \int_{2+p\mathbb{Z}_p} \mathrm{d}x\, \chi(-x\mathfrak{p}). \tag{62}$$

The corresponding coherent states $|\mathfrak{a}, b; s\rangle \equiv D(\mathfrak{a}, b, 0)|s\rangle$ have the wavefunction

$$\sigma(x; \mathfrak{a}, b; s) = \chi\left(-\tfrac{1}{2}\mathfrak{a}b + \mathfrak{a}x\right) s(x - b). \tag{63}$$

3.6. A linear map from $\mathcal{H}[\mathbb{Q}_p \times \mathbb{Q}_p]$ to $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$

Most of the literature on quantum mechanics on p-adic numbers considers systems with positions and momenta in $\mathbb{Q}_p$ which in our notation we denote as $S[\mathbb{Q}_p \times \mathbb{Q}_p]$. In this subsection we introduce a linear map from the Hilbert space $\mathcal{H}[\mathbb{Q}_p \times \mathbb{Q}_p]$ of the system $S[\mathbb{Q}_p \times \mathbb{Q}_p]$ to the Hilbert space $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$. These two Hilbert spaces are not isomorphic and the map that we introduce is not bijective. However it establishes a relationship between these two systems. From a practical point of view there is a lot of work on various special functions in $\mathcal{H}[\mathbb{Q}_p \times \mathbb{Q}_p]$ (e.g., in [24]) which through this map could be transferred into $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$.

We can express the fact that $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$ as periodicity of the functions $\tilde{g}(\mathfrak{p})$ (see also the appendix):

$$\tilde{g}(\mathfrak{p}) = \tilde{g}(\mathfrak{p} + 1). \tag{64}$$

In this sense there is some analogy between the present work and quantum mechanics on a circle. The map that we introduce in this subsection is similar to the Zak or Weil transform [9, 29] which maps functions on the real line to functions on a circle (e.g., Gaussians to Theta functions).

We denote with $\mathfrak{G}(x)$ where $x \in \mathbb{Q}_p$ the functions in the Hilbert space $\mathcal{H}[\mathbb{Q}_p \times \mathbb{Q}_p]$, and with $\tilde{\mathfrak{G}}(\varpi)$ their Fourier transform

$$\tilde{\mathfrak{G}}(\varpi) = \int_{\mathbb{Q}_p} \mathrm{d}x\, \chi(-x\mathfrak{p})\mathfrak{G}(x); \qquad \varpi \in \mathbb{Q}_p. \tag{65}$$

We express ϖ as

$$\begin{aligned} \varpi &= \sum_{\nu=\mathrm{ord}(\varpi)}^{\infty} \overline{\varpi}_\nu p^\nu = \mathfrak{p} + \alpha \\ \mathfrak{p} &= \sum_{\nu=\mathrm{ord}(\varpi)}^{-1} \overline{\varpi}_\nu p^\nu; \qquad \alpha = \sum_{\nu=0}^{\infty} \overline{\varpi}_\nu p^\nu \end{aligned} \tag{66}$$

and define a linear map from $\mathcal{H}[\mathbb{Q}_p \times \mathbb{Q}_p]$ to $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ as follows:

$$\tilde{\mathfrak{G}}(\varpi) = \tilde{\mathfrak{G}}(\mathfrak{p} + \alpha) \to \tilde{g}(\mathfrak{p}) = \int_{\mathbb{Z}_p} \mathrm{d}\alpha\, \tilde{\mathfrak{G}}(\mathfrak{p} + \alpha). \tag{67}$$

This map is such that the function $\tilde{g}(\mathfrak{p})$ defined through equation (67) obeys equation (64). We then prove that the inverse Fourier transform $\mathfrak{G}(x)$ of $\tilde{\mathfrak{G}}(\varpi)$ is mapped into the inverse Fourier transform $g(x)$ of $\tilde{g}(\mathfrak{p})$:

$$\begin{aligned}\mathfrak{G}(x) &= \int_{\mathbb{Q}_p} \mathrm{d}x\, \chi(x\varpi)\tilde{\mathfrak{G}}(\varpi) \to \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p} \int_{\mathbb{Z}_p} \mathrm{d}\alpha\, \chi(x\mathfrak{p} + x\alpha)\tilde{g}(\mathfrak{p}) \\ &= \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\, \chi(x\mathfrak{p})\tilde{g}(\mathfrak{p}) = g(x).\end{aligned} \tag{68}$$

The proof is based on the fact that $\chi(x\alpha) = 1$ for $x, \alpha \in \mathbb{Z}_p$.

4. Quantum systems with positions and momenta in the ring $\mathbb{Z}(p)$

We first discuss briefly some aspects of the theory of quantum systems with finite Hilbert space (reviewed in [10]) in order to define various quantities which we use later.

We consider a $S[\mathbb{Z}(p) \times \mathbb{Z}(p)]$ quantum system where positions and momenta take values in the field $\mathbb{Z}(p)$. The Hilbert space $\mathcal{H}[\mathbb{Z}(p) \times \mathbb{Z}(p)]$ of this system is p-dimensional. It comprises complex functions $f(m)$ where $m \in \mathbb{Z}(p)$, with the scalar product

$$\langle f|g\rangle = \sum_{m\in\mathbb{Z}(p)} [f(m)]^* g(m). \tag{69}$$

For convinience we normalize these functions to one ($\langle f|f\rangle = 1$).

In this Hilbert space, we consider an orthonormal basis which we call position states and denote as $|X; m\rangle$ where $m \in \mathbb{Z}(p)$. Here X is not a variable, but it simply indicates position states in $\mathcal{H}[\mathbb{Z}(p) \times \mathbb{Z}(p)]$. The resolution of the identity in terms of them is

$$\sum_{m\in\mathbb{Z}(p)} |X; m\rangle\langle X; m| = \mathbf{1}. \tag{70}$$

The

$$\omega(m) = \exp\left(\frac{\mathrm{i}2\pi m}{p}\right); \qquad \frac{1}{p}\sum_m \omega(mn) = \delta(n, 0); \qquad m, n \in \mathbb{Z}(p), \tag{71}$$

where $\delta(n, 0)$ is the Kronecker delta are additive characters. Using them we define the Fourier operator,

$$F = p^{-1/2} \sum_{m,n\in\mathbb{Z}(p)} \omega(mn)|X; m\rangle\langle X; n|; \qquad F^4 = \mathbf{1}. \tag{72}$$

Acting with the Fourier operator on the position states we get momentum states which form another orthonormal basis

$$|P; m\rangle = F|X; m\rangle = p^{-1/2} \sum_{n\in\mathbb{Z}(p)} \omega(mn)|X; n\rangle. \tag{73}$$

Here P is not a variable but it simply indicates momentum states in $\mathcal{H}[\mathbb{Z}(p) \times \mathbb{Z}(p)]$.

We also introduce the position and momentum operators

$$\widehat{X} = \sum_{n=0}^{p-1} n|X; n\rangle\langle X; n|; \qquad \widehat{P} = F\widehat{X}F^\dagger = \sum_{n=0}^{p-1} n|P; n\rangle\langle P; n|. \tag{74}$$

In the summation we have specified that n takes values from 0 to $p - 1$ and this ensures that these operators are single-valued. If we sum from kp up to $(k + 1)p - 1$ then we get these operators plus $k\mathbf{1}$.

4.1. *The Heisenberg–Weyl group $HW[\mathbb{Z}(p) \times \mathbb{Z}(p)]$*

The position-momentum phase space of this system is the toroidal lattice $\mathbb{Z}(p) \times \mathbb{Z}(p)$. In this phase space we define the displacement operators

$$\begin{aligned} Z(\alpha) &= \sum_{n\in\mathbb{Z}(p)} \omega(n\alpha)|X;n\rangle\langle X;n| = \omega(\alpha\widehat{X}), \\ X(\beta) &= F^{\dagger}Z(\beta)F = \sum_{n\in\mathbb{Z}(p)} \omega(-n\beta)|P;n\rangle\langle P;n| = \omega(-\beta\widehat{P}), \end{aligned} \tag{75}$$

where $\alpha, \beta \in \mathbb{Z}(p)$. Acting with them on momentum and position states we get

$$Z(\alpha)|P;m\rangle = |P;m+\alpha\rangle; \qquad X(\beta)|X;m\rangle = |X;m+\beta\rangle. \tag{76}$$

The displacement operators obey the relations

$$X(\beta)Z(\alpha) = Z(\alpha)X(\beta)\omega(-\alpha\beta), \qquad \alpha, \beta \in \mathbb{Z}(p). \tag{77}$$

General displacement operators are given by

$$D(\alpha, \beta, \gamma) = Z(\alpha)X(\beta)\omega(\gamma - 2^{-1}\alpha\beta); \qquad \alpha, \beta, \gamma \in \mathbb{Z}(p), \tag{78}$$

$2^{-1} = (p+1)/2$ is the inverse of 2 within $\mathbb{Z}(p)$, where p is an odd prime number.

We can prove that

$$D(\alpha, \beta, \gamma)D(\alpha, \beta, \gamma) = D[\alpha+\alpha', \beta+\beta', \gamma+\gamma'+2^{-1}(\alpha\beta'-\alpha'\beta)], \tag{79}$$

and therefore the operators $D(\alpha, \beta, \gamma)$ form a representation of the Heisenberg–Weyl group $HW[\mathbb{Z}(p) \times \mathbb{Z}(p)]$.

4.2. *Prüfer operators in the system $S[\mathbb{Z}(p) \times \mathbb{Z}(p)]$*

The Prüfer operators are generalizations of the displacement operators $Z(\alpha)$ and $X(\beta)$. They are unitary $p \times p$ matrices defined as

$$\begin{aligned} A_P(\mathfrak{p}) &= \sum_{m=0}^{p-1} \chi(-\mathfrak{p}m)|P;m\rangle\langle P;m| = \chi(-\mathfrak{p}\widehat{P}); \qquad \mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p \\ A_X(\mathfrak{p}) &= FA_P(\mathfrak{p})F^{\dagger} = \sum_{m=0}^{p-1} \chi(\mathfrak{p}m)|X;m\rangle\langle X;m| = \chi(\mathfrak{p}\widehat{X}). \end{aligned} \tag{80}$$

In the summation we have specified that m takes values from 0 to $p-1$ and this ensures that these operators are single-valued. For example, if we sum from p up to $2p-1$ then we get these operators times the phase factor $\chi(-\mathfrak{p}p)$. The sums in equations (80) can be viewed as p-adic integrals on $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}(p)$. For example,

$$A_P(\mathfrak{p}) = p\int_{\mathbb{Z}_p/p\mathbb{Z}_p} \mathrm{d}x\, \chi(\mathfrak{p}x)|P;x\rangle\langle P;x|. \tag{81}$$

It is easily seen that

$$A_P(\mathfrak{p})A_P(\mathfrak{p}') = A_P(\mathfrak{p}+\mathfrak{p}'); \qquad A_P(0) = \mathbf{1}. \tag{82}$$

Therefore the operators $A_P(\mathfrak{p})$ form a group $\mathfrak{A}_P$ which is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$ and also to the Prüfer p-group:

$$\mathfrak{A}_P \equiv \{A_P(\mathfrak{p})|\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p\} \cong \mathbb{Q}_p/\mathbb{Z}_p \cong C(p^{\infty}). \tag{83}$$

The same is true for the operators $A_X(\mathfrak{p})$:

$$\mathfrak{A}_X \equiv \{A_X(\mathfrak{p})|\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p\} \cong \mathbb{Q}_p/\mathbb{Z}_p \cong C(p^{\infty}). \tag{84}$$

We next consider an element of $\mathbb{Q}_p/\mathbb{Z}_p$

$$\mathfrak{p} = \overline{\mathfrak{p}}_{-k}p^{-k} + \overline{\mathfrak{p}}_{-k+1}p^{-k+1} + \cdots + \overline{\mathfrak{p}}_{-1}p^{-1}; \qquad k = -\mathrm{ord}(\mathfrak{p}); \qquad 0 \leqslant \overline{\mathfrak{p}}_i \leqslant p-1 \tag{85}$$

and use the notation

$$\begin{aligned} \mathfrak{p} &= \mathfrak{L}(\mathfrak{p}) + \overline{\mathfrak{p}}_{-1}p^{-1} \\ \mathfrak{L}(\mathfrak{p}) &= \overline{\mathfrak{p}}_{-k}p^{-k} + \cdots + \overline{\mathfrak{p}}_{-2}p^{-2}. \end{aligned} \tag{86}$$

If $\mathrm{ord}(\mathfrak{p}) = -1$, then $\mathfrak{L}(\mathfrak{p}) = 0$. Inserting equation (86) into equation (80) we prove that

$$\begin{aligned} A_P(\mathfrak{p}) &= A_P[\mathfrak{L}(\mathfrak{p})]X[\overline{\mathfrak{p}}_{-1}], \\ A_X(\mathfrak{p}) &= A_X[\mathfrak{L}(\mathfrak{p})]Z[\overline{\mathfrak{p}}_{-1}]. \end{aligned} \tag{87}$$

Proposition 4.1.

(1)

$$p^{-1}\int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\, A_P(\mathfrak{p}) = |P;0\rangle\langle P;0|. \tag{88}$$

(2) Let Θ be an operator acting on $\mathcal{H}[\mathbb{Z}(p)\times\mathbb{Z}(p)]$. Then

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\, A_P(\mathfrak{p})\Theta[A_P(\mathfrak{p})]^\dagger = [\mathrm{tr}\,\Theta]\mathbf{1}. \tag{89}$$

Similar properties are also valid for $A_X(\mathfrak{p})$.

Proof. Integration of equation (81) with respect to $\mathfrak{p}$, taking into account equation (18), proves the first part of the proposition.

For the second part, we insert equation (81) into equation (89), and perform the integration taking into account equation (18). □

Properties analogous to equations (88) and (89) are also true for the displacement operators in general $S[\mathbb{Z}(d)\times\mathbb{Z}(d)]$ systems (see [10]). This reconfirms the role of the Prüfer operators as generalized displacement operators.

The phase space of the $S[\mathbb{Z}(p)\times\mathbb{Z}(p)]$ system is the toroidal lattice $\mathbb{Z}(p)\times\mathbb{Z}(p)$. The Prüfer operators introduce a finer grid into this lattice. We note however that there is no simple relation between $A_P(\mathfrak{p})A_X(\mathfrak{p}')$ and $A_X(\mathfrak{p}')A_P(\mathfrak{p})$ analogous to equation (77), and therefore this is a weaker formalism.

4.3. Prüfer states in the system $S[\mathbb{Z}(p)\times\mathbb{Z}(p)]$

Let $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$. The Prüfer states are defined as

$$\begin{aligned} |A_X;\mathfrak{p}\rangle &\equiv A_P(\mathfrak{p})|X;0\rangle = p^{-1/2}\sum_{m=0}^{p-1}\chi(-\mathfrak{p}m)|P;m\rangle = A_P[\mathfrak{L}(\mathfrak{p})]|X;\overline{\mathfrak{p}}_{-1}\rangle \\ |A_P;\mathfrak{p}\rangle &\equiv A_X(\mathfrak{p})|P;0\rangle = p^{-1/2}\sum_{m=0}^{p-1}\chi(\mathfrak{p}m)|X;m\rangle = A_X[\mathfrak{L}(\mathfrak{p})]|P;\overline{\mathfrak{p}}_{-1}\rangle, \end{aligned} \tag{90}$$

where the A_P, A_X in the notation of these states are not variables but they simply indicate the two types of Prüfer states. In the case $\mathrm{ord}(\mathfrak{p}) = -1$ the Prüfer states are position and momentum states.

Proposition 4.2.

(1) The Prüfer states obey the resolution of the identity

$$\int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}|A_P;\mathfrak{p}\rangle\langle A_P;\mathfrak{p}| = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}|A_X;\mathfrak{p}\rangle\langle A_X;\mathfrak{p}| = \mathbf{1}. \tag{91}$$

(2)

$$|\langle P;\overline{\mathfrak{p}}_{-1}|A_P;\mathfrak{p}\rangle| = |\langle X;\overline{\mathfrak{p}}_{-1}|A_X;\mathfrak{p}\rangle| = \frac{\sin[p\pi\mathfrak{L}(\mathfrak{p})]}{p\sin[\pi\mathfrak{L}(\mathfrak{p})]}. \tag{92}$$

Proof. For the first part we use equation (89) with $\Theta = |X;0\rangle\langle X;0|$ and $\Theta = |P;0\rangle\langle P;0|$.

For the second part we use the relation

$$\sum_{k=-N}^{N} \exp(ik\theta) = \frac{\sin\left[(2N+1)\frac{\theta}{2}\right]}{\sin\left[\frac{\theta}{2}\right]} \tag{93}$$

with $\theta = 2\pi\mathfrak{L}(\mathfrak{p})$ and $2N+1 = p$. □

For small $\pi\mathfrak{L}(\mathfrak{p})$ equation (92) shows that the Prüfer states are very close to the position and momentum states. In this case the Prüfer operators are very close to the displacement operators, and in equation (87), the operators $A_P[\mathfrak{L}(\mathfrak{p})]$, $A_X[\mathfrak{L}(\mathfrak{p})]$ are small corrections:

$$\begin{aligned} &|A_P;\mathfrak{p}\rangle \approx |P;\overline{\mathfrak{p}}_{-1}\rangle; \qquad &|A_X;\mathfrak{p}\rangle \approx |X;\overline{\mathfrak{p}}_{-1}\rangle \\ &A_P(\mathfrak{p}) \approx X(\overline{\mathfrak{p}}_{-1}); \qquad &A_X(\mathfrak{p}) \approx Z(\overline{\mathfrak{p}}_{-1}). \end{aligned} \tag{94}$$

5. A $S[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ system as semi-infinite chain of coupled $S[\mathbb{Z}(p) \times \mathbb{Z}(p)]$ subsystems

In this section we discuss the 'quantum-engineering' of an $S[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ system, from a semi-infinite chain of $S[\mathbb{Z}(p) \times \mathbb{Z}(p)]$ systems (e.g., spins with $j = (p-1)/2$), which are coupled in a particular way. This special coupling will give p-adic structure to this semi-infinite chain.

We consider a semi-infinite chain of $S[\mathbb{Z}(p) \times \mathbb{Z}(p)]$ systems with Hilbert space

$$\mathcal{H}_{\text{chain}} = \mathcal{H}[\mathbb{Z}(p) \times \mathbb{Z}(p)] \otimes \mathcal{H}[\mathbb{Z}(p) \times \mathbb{Z}(p)] \otimes \cdots \tag{95}$$

and define a linear map $\mathfrak{R}$ between the states in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ and the states in $\mathcal{H}_{\text{chain}}$. The position basis in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ is mapped to the position basis in $\mathcal{H}_{\text{chain}}$, as follows:

$$\begin{aligned} &|\mathsf{X};x\rangle \to |\mathsf{X}^{\text{chain}};x\rangle = [p^{1/2}|X;\overline{x}_0\rangle] \otimes [p^{1/2}|X;\overline{x}_1\rangle] \otimes [p^{1/2}|X;\overline{x}_2\rangle] \otimes \cdots. \\ &x = \overline{x}_0 + \overline{x}_1 p + \overline{x}_2 p^2 + \cdots. \end{aligned} \tag{96}$$

The factors $p^{1/2}$ on the right-hand side effectively introduce the Haar measure in scalar products in $\mathcal{H}_{\text{chain}}$. In order to see this explicitly, we consider the resolution of the identity in terms of the position states in the two Hilbert spaces. The left-hand side of the resolution of the identity in equation (29) can be written as

$$\begin{aligned} \int_{\mathbb{Z}_p} |\mathsf{X};x\rangle\langle\mathsf{X};x|\,\mathrm{d}x = \lim_{\ell\to\infty} p^{-\ell} \sum |X;\overline{x}_0 + \overline{x}_1 p + \ldots + \overline{x}_{\ell-1}p^{\ell-1}\rangle \\ \times\langle X;\overline{x}_0 + \overline{x}_1 p + \ldots + \overline{x}_{\ell-1}p^{\ell-1}|, \end{aligned} \tag{97}$$

where the summation is over all $\overline{x}_k$. If we replace the states in equation (97) with their counterparts in the chain given in equation (96), the normalization factors $p^{1/2}$ are needed in order to cancel $p^{-\ell}$ so that we get the identity $\mathbf{1}$. A different way of saying this is that the states $|\mathsf{X}; x\rangle$ are normalized to a delta function, while the states $|X; \overline{x}_i\rangle$ are normalized to 1. The appearance of an infinite number of $p^{1/2}$ factors on the right-hand side of equation (96) is related to the fact that the position states belong to the corresponding rigged Hilbert space.

The Hilbert spaces $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ and $\mathcal{H}_{\text{chain}}$ are isomorphic to each other. Below we study explicitly the correspondence of various states and operators in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ with their counterparts in $\mathcal{H}_{\text{chain}}$.

Locally constant functions of degree ℓ in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$, correspond to functions $f(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{\ell-1})$ in $\mathcal{H}_{\text{chain}}$, i.e., to functions which do not depend on $\overline{x}_\ell, \overline{x}_{\ell+1}, \ldots$.

5.1. Fourier operators

We substitute equation (96) into the Fourier operator F_ℓ of equation (41) which acts on the subspace $\mathcal{H}_\ell$, and we get

$$\mathsf{F}_\ell \to \mathsf{F}_\ell^{\text{chain}} = p^{-\ell/2} \sum \chi \left(\sum_{i,j} \overline{x}_i \overline{y}_j g_{ij}^{(\ell)} \right) |X; \overline{x}_0\rangle\langle X; \overline{y}_0| \otimes \cdots \otimes |X; \overline{x}_{\ell-1}\rangle\langle X; \overline{y}_{\ell-1}|, \tag{98}$$

where the summation is over all $\overline{x}_i, \overline{y}_i$ such that $0 \leqslant \overline{x}_i, \overline{y}_i \leqslant p-1$. The 'coupling matrix' $\left(g_{ij}^{(\ell)}\right)$ is a $\ell \times \ell$ symmetric matrix, given by

$$\begin{aligned} g_{ij}^{(\ell)} &= p^{i+j-\ell} \qquad \text{if} \quad i+j \leqslant \ell - 1 \\ g_{ij}^{(\ell)} &= 0 \qquad \text{if} \quad i+j > \ell - 1. \end{aligned} \tag{99}$$

This matrix effectively rescales the $(\overline{y}_0, \overline{y}_1, \overline{y}_2, \ldots)$ on the right-hand side of equation (98), which is an element of $\mathbb{Z}(p) \times \mathbb{Z}(p) \times \ldots$, into $(\overline{y}_0, p\overline{y}_1, p^2\overline{y}_2, \ldots)$ which is an element of $\mathbb{Z}_p$. Similarly, it rescales the $(\overline{x}_0, \overline{x}_1, \overline{x}_2, \ldots)$ which is an element of $\mathbb{Z}(p) \times \mathbb{Z}(p) \times \ldots$, into $(p^{-\ell}\overline{x}_0, p^{-\ell+1}\overline{x}_1, p^{-\ell+2}\overline{x}_2, \ldots)$ which is an element of $\mathbb{Q}_p/\mathbb{Z}_p$. Of course, we need to check that these numbers have the p-adic addition and multiplication rule. We discuss this in detail below, where we act with displacement operators (as implemented in the chain) on position and momentum states.

$\mathsf{F}_\ell^{\text{chain}}$ is different from the Fourier operator $F \otimes F \otimes \cdots \otimes F$ (where F has been given in equation (72)) which performs independent Fourier transforms on each subsystem in the chain. The latter Fourier transform can be written in the form of equation (98) with the matrix $g_{ij}^{(\ell)}$ replaced by the unit matrix $\mathbf{1}_\ell$.

5.2. Displacement operators

Proposition 5.1. *The linear map $\mathfrak{R}$ implies the following correspondence between states and operators in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$, and their counterparts in $\mathcal{H}_{\text{chain}}$:*

(1)

$$\widehat{\mathsf{X}} \to \widehat{\mathsf{X}}^{\text{chain}} = \left[(p\widehat{X}) \otimes (p\mathbf{1}) \otimes (p\mathbf{1}) \otimes \cdots\right] + p\left[(p\mathbf{1}) \otimes (p\widehat{X}) \otimes (p\mathbf{1}) \otimes \cdots\right] + \cdots. \tag{100}$$

(2) The momentum states in $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ correspond to Prüfer states in $\mathcal{H}_{\text{chain}}$

$$|\mathsf{P}; \mathfrak{p}\rangle \to |\mathsf{P}^{\text{chain}}; \mathfrak{p}\rangle = |A_P; \mathfrak{p}\rangle \otimes |A_P; p\mathfrak{p}\rangle \otimes \cdots \otimes |A_P; p^{k-1}\mathfrak{p}\rangle \otimes |P; 0\rangle \otimes \cdots, \tag{101}$$

where $k = -\mathrm{ord}(\mathfrak{p})$. *If* $\pi\mathfrak{L}(\mathfrak{p})$ *is small, then approximately*

$$|\mathsf{P}^{\text{chain}}; \mathfrak{p}\rangle \approx |P; \overline{\mathfrak{p}}_{-1}\rangle \otimes |P; \overline{\mathfrak{p}}_{-2}\rangle \otimes \cdots \otimes |P; \overline{\mathfrak{p}}_{-k}\rangle \otimes |P; 0\rangle \otimes \cdots, \tag{102}$$

where $\mathfrak{p} = \overline{\mathfrak{p}}_{-k} p^{-k} + \cdots + \overline{\mathfrak{p}}_{-1} p^{-1}$.

(3) $\widehat{\mathsf{P}} \to \widehat{\mathsf{P}}^{\text{chain}}$ *where approximately*

$$\widehat{\mathsf{P}}^{\text{chain}} \approx p^{-1}[\widehat{P} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots] + p^{-2}[\mathbf{1} \otimes \widehat{P} \otimes \mathbf{1} \otimes \cdots] + \cdots. \tag{103}$$

(4) *The displacement operator* $\mathsf{Z}(\mathfrak{a})$ *corresponds to a product of Prüfer operators in* $\mathcal{H}_{\text{chain}}$*:*

$$\mathsf{Z}(\mathfrak{a}) \to \mathsf{Z}(\mathfrak{a})^{\text{chain}} = A_X(\mathfrak{a}) \otimes A_X(\mathfrak{a}p) \otimes \cdots \otimes A_X(\mathfrak{a}p^{k-1}) \otimes \mathbf{1} \otimes \cdots, \tag{104}$$

where $k = -\mathrm{ord}(\mathfrak{a})$. *An approximate but simpler expression is*

$$\mathsf{Z}(\mathfrak{a})^{\text{chain}} \approx Z(\overline{\mathfrak{a}}_{-1}) \otimes Z(\overline{\mathfrak{a}}_{-2}) \otimes \cdots \otimes Z(\overline{\mathfrak{a}}_{-k}) \otimes \mathbf{1} \otimes \cdots, \tag{105}$$

where $\mathfrak{a} = \overline{\mathfrak{a}}_{-k} p^{-k} + \cdots + \overline{\mathfrak{a}}_{-1} p^{-1}$.

(5) *The displacement operator* $\mathsf{X}(b)$ *corresponds approximately to a product of Prüfer operators in* $\mathcal{H}_{\text{chain}}$*:*

$$\mathsf{X}(b) \to \mathsf{X}(b)^{\text{chain}} \approx A_P(bp^{-1}) \otimes A_P(bp^{-2}) \otimes \cdots, \tag{106}$$

where $b = \overline{b}_0 + \overline{b}_1 p + \cdots$ *and* $bp^{-i} = \overline{b}_0 p^{-i} + \cdots + \overline{b}_{i-1} p^{-1} \in \mathbb{Q}_p/\mathbb{Z}_p$. *A weaker but simpler approximation is*

$$\mathsf{X}(b)^{\text{chain}} \approx X(\overline{b}_0) \otimes X(\overline{b}_1) \otimes \cdots \tag{107}$$

Proof. Substitution of equation (96) into equations (35) proves the first part of the proposition.

For $k = -\mathrm{ord}(\mathfrak{p})$ and $x = \overline{x}_0 + \overline{x}_1 p + \overline{x}_2 p^2 + \cdots$ we easily see that

$$\chi(x\mathfrak{p}) = \chi(\overline{x}_0\mathfrak{p})\chi(\overline{x}_1 p\mathfrak{p}) \ldots \chi(\overline{x}_{k-1} p^{k-1}\mathfrak{p}) \tag{108}$$

and then using equation (90) we prove the second part of the proposition. The approximate expression of equation (102) is proved using equation (94).

Using equation (102) in conjunction with equation (40) we prove the third part of the proposition. The fourth part of the proposition is proved using equations (43) and (100). Combining equation (103) with equation (43) we prove the fifth part of the proposition. □

There is no simple exact expression for the momentum operator $\widehat{\mathsf{P}}^{\text{chain}}$ in terms of the momentum operators in the subsystems; or for the displacement operator $X(b)^{\text{chain}}$ in terms of the displacement operators in the subsystems. But we have given approximate expressions for these quantities.

For $\mathsf{Z}(\mathfrak{a})^{\text{chain}}$ we have exact relations and easily show that

$$\mathsf{Z}(\mathfrak{a})^{\text{chain}}|\mathsf{P}^{\text{chain}}; \mathfrak{p}\rangle = |\mathsf{P}^{\text{chain}}; \mathfrak{p} + \mathfrak{a}\rangle; \qquad \mathsf{Z}(\mathfrak{a})^{\text{chain}}|\mathsf{X}^{\text{chain}}; x\rangle = \chi(\mathfrak{a}x)|\mathsf{X}^{\text{chain}}; x\rangle. \tag{109}$$

For $\mathsf{X}(b)^{\text{chain}}$ we have approximate relations and will use equation (107) which is a weaker approximation but easier to use. We show that

$$\begin{gathered} \mathsf{X}(b)^{\text{chain}}|\mathsf{X}^{\text{chain}}; x\rangle = |\mathsf{X}^{\text{chain}}; c\rangle \\ x = \overline{x}_0 + \overline{x}_1 p + \cdots; b = \overline{b}_0 + \overline{b}_1 p + \cdots; c = \overline{c}_0 + \overline{c}_1 p + \cdots, \end{gathered} \tag{110}$$

where $c_i = x_i + b_i (\mathrm{mod}\ p)$. The exact result should have c equal to the p-adic sum $x + b$. Here c may not be equal to $x + b$ because it does not have the 'carrying' of p-adic addition. This is the error with the approximation in equation (107), for this relation.

We also act with $\mathsf{X}(b)^{\text{chain}}$ (of equation (107)) on momentum states, using not the exact relation of equation (101) but the approximation of equation (102):

$$\begin{aligned}&\mathsf{X}(b)^{\text{chain}}|\mathsf{P}^{\text{chain}};\mathfrak{p}\rangle = \exp(\mathrm{i}2\pi\theta)|\mathsf{P}^{\text{chain}};\mathfrak{p}\rangle\\ &b=\overline{b}_0+\overline{b}_1 p+\cdots; \qquad \mathfrak{p}=\overline{\mathfrak{p}}_{-k}p^{-k}+\overline{\mathfrak{p}}_{-k+1}p^{-k+1}+\cdots+\overline{\mathfrak{p}}_{-1}p^{-1}\\ &\theta=\overline{b}_0\overline{\mathfrak{p}}_{-1}+\overline{b}_1\overline{\mathfrak{p}}_{-2}+\overline{b}_2\overline{\mathfrak{p}}_{-3}+\cdots.\end{aligned} \tag{111}$$

The exact result should have a phase factor $\chi(b\mathfrak{p})$ and this has extra terms in addition to those appearing in θ. It is seen that here the combined approximation of equations (107) and (102) fails to give all the terms in the p-adic multiplication.

The conclusion is that the exact operators like $\mathsf{Z}(\mathfrak{a})^{\text{chain}}$ produce results based on the p-adic adddition and multiplication rule. Approximate expressions like $\mathsf{X}(b)^{\text{chain}}$ of equation (107) introduce errors.

5.3. Hamiltonian and coupling between the subsystems

A state $|f\rangle$ in $\mathcal{H}[\mathbb{Z}_p\times(\mathbb{Q}_p/\mathbb{Z}_p)]$ evolves in time as $\exp[\mathrm{i}tH(\widehat{\mathsf{X}},\widehat{\mathsf{P}})]|f\rangle$ where $H(\widehat{\mathsf{X}},\widehat{\mathsf{P}})$ is the Hamiltonian of the system. In our discussion below we use the space $\mathcal{H}_\ell$ of locally constant wavefunctions with degree ℓ. There is no loss of generality in doing this, because all functions in $\mathcal{H}[\mathbb{Z}_p\times(\mathbb{Q}_p/\mathbb{Z}_p)]$ are locally constant. The advantage is that position and momentum operators are well defined and we avoid the use of rigged Hilbert spaces. In this case it is sufficient to use the Hamiltonian

$$\Pi_\ell H(\widehat{\mathsf{X}},\widehat{\mathsf{P}})\Pi_\ell = H(\widehat{\mathsf{X}}_\ell,\widehat{\mathsf{P}}_\ell) = H\left(\widehat{\mathsf{X}}_\ell,\mathsf{F}_\ell\widehat{\mathsf{X}}_\ell\mathsf{F}_\ell^\dagger\right). \tag{112}$$

The system S_{chain} evolves in time in the same way as the $S[\mathbb{Z}_p\times(\mathbb{Q}_p/\mathbb{Z}_p)]$ if its Hamiltonian is of the form

$$H\left(\widehat{\mathsf{X}}_\ell^{\text{chain}},\widehat{\mathsf{P}}_\ell^{\text{chain}}\right) = H\left(\widehat{\mathsf{X}}_\ell^{\text{chain}},\mathsf{F}_\ell^{\text{chain}}\widehat{\mathsf{X}}_\ell^{\text{chain}}\mathsf{F}_\ell^{\text{chain}\dagger}\right). \tag{113}$$

The Fourier transform $\mathsf{F}_\ell^{\text{chain}}$ of equation (98), which contains the coupling matrix $g^{(\ell)}$, enters here. Physically, these Hamiltonians describe chains with a special coupling between their components. This special coupling is related to the matrix $g^{(\ell)}$ which as we explained earlier, rescales the positions $(\overline{x}_0,\overline{x}_1,\overline{x}_2,\ldots)$ in the chain into $(\overline{x}_0,p\overline{x}_1,p^2\overline{x}_2,\ldots)$ which is a p-adic integer. We discussed earlier how the p-adic addition and multiplication rules are implemented in the chain.

The set of Hamiltonians in equation (113) is a subset of the more general Hamiltonians

$$H(\widehat{X}_0,\widehat{P}_0;\widehat{X}_1,\widehat{P}_1;,,,) = H(\widehat{X}_0,F\widehat{X}_0F^\dagger;\widehat{X}_1,F\widehat{X}_1F^\dagger;\ldots) \tag{114}$$

which describes general S_{chain} systems coupled in an arbitrary way. For clarity we used here indices in the position and momentum operators, which indicate the subsystem within the chain, on which they act.

5.4. Topological structure

In this subsection we discuss how the semi-infinite chain of spins acquires the totally disconnected topology associated with the p-adic integers.

We consider the p-adic integer $\overline{x}_0+\overline{x}_1p+\cdots$ (where $0\leqslant\overline{x}_i\leqslant p-1$). We refer to $\overline{x}_i$ as coordinate with hierarchy i, in the sense that the ultrametric distance between two p-adic integers with the same $(\overline{x}_0,\ldots,\overline{x}_{i-1})$ and different $\overline{x}_i$ is p^{-i} (large i represents low hierarchy in the sense that the ultrametric distance is very small).

In S_{chain} the Hilbert space of the ith subsystem is spanned by the position states $|X;\overline{x}_i\rangle$ and for this reason we call it subsystem with hierarchy i. The hierarchy of the subsystems is created by the coupling matrix $g^{(\ell)}$ which as we explained earlier produces an i-dependent rescaling of $\overline{x}_i$ into $\overline{x}_i p^i$.

Equation (102) shows that the momentum states in this subsystem are approximately $|P; \overline{\mathfrak{p}}_{-i-1}\rangle$. Therefore as we move towards the right of the chain, the hierarchy of the subsystems decreases, the corresponding positions $\overline{x}_i p^i$ become very small and the corresponding momenta $\overline{\mathfrak{p}}_{-i-1} p^{-i-1}$ become very large (in ultrametric). Position displacements limited to low hierarchy subsystems

$$\mathsf{X}(b) \approx \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes X(\overline{b}_i) \otimes X(\overline{b}_{i+1}) \otimes \cdots \qquad b = \overline{b}_i p^i + \overline{b}_{i+1} p^{i+1} + \cdots \tag{115}$$

are close to $\mathbf{1}$ and this is related to the totally disconnected topology of the profinite group K and the neighbourhoods in equation (49). In contrast, momentum displacements limited to low hierarchy subsystems are large (see equation (105)) and this is related to discrete topology of the group G.

6. Discussion

We have considered a $S[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ system and studied displacement operators and the corresponding Heisenberg–Weyl group $HW[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$. We have also considered coherent states in this context and used them to define a Bargmann-like representation.

Such a system can be engineered from a semi-infinite chain of $j = (p-1)/2$ spins which are coupled as described by the Hamiltonian of equation (113) and in particular by the non-diagonal elements of the coupling matrix in equation (99) which is intimately connected to p-adic theory. We have studied in detail the isomorphism between the Hilbert spaces $\mathcal{H}[\mathbb{Z}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)]$ and $\mathcal{H}_{\text{chain}}$. We have also discussed the mechanism which gives the chain the totally disconnected topology associated with p-adic integers.

The work introduces ideas from number theory to quantum mechanics and harmonic analysis.

Appendix

Let $f(\mathfrak{p})$ where $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$ be a complex function with compact support. We have explained earlier that we can regard it as a function $f(r)$ where $r \in \mathbb{Q}_p$, which is periodic:

$$f(r) = f(r+1) \tag{A.1}$$

r can be written as $r = \mathfrak{p} + x$ where $\mathfrak{p} \in \mathbb{Q}_p/\mathbb{Z}_p$ and $x \in \mathbb{Z}_p$. Strictly speaking $\mathfrak{p}$ is here the element of the coset with zero integral part; but we for simplicity we use the same notation for the two.

Integration of $f(r)$ over $\mathbb{Q}_p$ can be written as

$$\int_{\mathbb{Q}_p} \mathrm{d}r\, f(r) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p} \int_{\mathbb{Z}_p} \mathrm{d}x\, f(\mathfrak{p}+x) = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\, f(\mathfrak{p}) \int_{\mathbb{Z}_p} \mathrm{d}x = \int_{\mathbb{Q}_p/\mathbb{Z}_p} \mathrm{d}\mathfrak{p}\, f(\mathfrak{p}). \tag{A.2}$$

The counting measure on $\mathbb{Q}_p/\mathbb{Z}_p$ ensures that the above relation holds (see also [25]).

Equation (A.2) can be regarded as a generalization of the Zak or Weil transform [29] which takes functions from the real line $\mathbb{R}$ to the circle $\mathbb{R}/\mathbb{Z}$. In this case the subgroup $\mathbb{Z}$ is discrete. In our case the subgroup $\mathbb{Z}_p$ is totally disconnected.

References

[1] Weyl H 1950 *Theory of Groups and Quantum Mechanics* (New York: Dover)
[2] Schwinger J 1960 *Proc. Natl Acad. Sci. USA* **46** 570
Schwinger J 1970 *Quantum Kinematics and Dynamics* (New York: Benjamin)
[3] Auslander L and Tolimieri R 1979 *Bull. Am. Math. Soc.* **1** 847
Hannay J and Berry M V 1980 *Physica* D **1** 267
[4] Stovicek P and Tolar J 1984 *Rep. Math. Phys.* **20** 157
Tolar J and Hadzitaskos G 1997 *J. Phys. A: Math. Gen.* **30** 2509
[5] Balian R and Itzykson C 1986 *C. R. Acad. Sci.* **303** 773
Mehta M L 1987 *J. Math. Phys.* **28** 781
Galetti D and de Toledo-Piza A F R 1988 *Physica* A **149** 267
Varadarajan V S 1995 *Lett. Math. Phys.* **34** 319
[6] Figueroa H, Gracia-Bondia J M and Varilly J C 1990 *J. Math. Phys.* **31** 2664
Varilly J C and Gracia-Bondia J M 1989 *Ann. Phys., NY* **190** 107
[7] Vourdas A 1990 *Phys. Rev.* A **41** 1653
Vourdas A 1991 *Phys. Rev.* A **43** 1564
Vourdas A and Bendjaballah C 1993 *Phys. Rev.* A **47** 3523
Vourdas A 1996 *J. Phys. A: Math. Gen.* **29** 4275
[8] Terras A 1997 *Fourier Analysis on Finite Groups and Applications* (Cambridge: Cambridge University Press)
[9] Weil A 1964 *Acta Math.* **111** 143
[10] Vourdas A 2004 *Rep. Prog. Phys.* **67** 267
[11] Neuhauser M 2002 *J. Lie Theory* **12** 15
Feichtinger H G, Hazewinkel M, Kaiblinger N, Matusiak E and Neuhauser M 2008 *Quantum J. Math.* **59** 15
[12] Vourdas A 2005 *J. Phys. A: Math. Gen.* **38** 8453
Vourdas A 2006 *Acta Appl. Math.* **93** 197
Vourdas A 2006 *J. Math. Phys.* **47** 092104
Vourdas A 2008 *J. Fourier Anal. Appl.* **14** 102
[13] Vourdas A 2007 *J. Phys. A: Math. Gen.* **40** R285
[14] Wootters W 1987 *Ann. Phys., NY* **176** 1
Wootters W and Fields B D 1989 *Ann. Phys., NY* **191** 363
Gibbons K, Hoffman M J and Wootters W 2004 *Phys. Rev.* A **70** 062101
[15] Albouy O and Kibler M 2007 *SIGMA* **3** 076
Kibler M R 2006 *Int. J. Mod. Phys.* B **20** 1792
Kibler M R 2006 *Int. J. Mod. Phys.* B **20** 1802
Kibler M R and Planat M 2006 *Int. J. Mod. Phys.* B **20** 1802
[16] Klimov A, Sanchez-Soto L and de Guise H 2005 *J. Phys. A: Math. Gen.* **38** 2747
Klimov A, Sanchez-Soto L and de Guise H 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** 283
Romero J L, Bjork G, Klimov A B and Sanchez-Soto L L 2005 *Phys. Rev.* A **72** 062310
Klimov A B, Munoz C and Romero J L 2006 *J. Phys. A: Math. Gen.* **39** 14471
[17] Saniga M, Planat M and Rosu H 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** L19
Planat M and Rosu H 2005 *Eur. Phys. J.* D **36** 133
Saniga M and Planat M 2006 *J. Phys. A: Math. Gen.* **39** 435
[18] Gouvea F Q 1993 *p-Adic Numbers: An Introduction* (Berlin: Springer)
Mahler K 1973 *Introduction to p-Adic Numbers and their Functions* (Cambridge: Cambridge University Press)
Robert A M 1993 *A Course in p-Adic Analysis* (Berlin: Springer)
[19] Khrennikov A 1991 *J. Math. Phys.* **32** 932
Albeverio S and Khrennikov A 1996 *J. Phys. A: Math. Gen.* **29** 5515
Albeverio S, Cianci R and Khrennikov A 1997 *J. Phys. A: Math. Gen.* **30** 881
Khrennikov A and Kozyrev S V 2006 *Physica* A **359** 222
[20] Brekke L, Freund P, Olson M and Witten E 1988 *Nucl. Phys.* B **302** 365
Freund P and Olson M 1988 *Nucl. Phys.* B **297** 86
Meurice Y 1991 *Commun. Math. Phys.* **135** 303
Ruelle Ph, Thiran E, Verstegen D and Weyers J 1989 *J. Math. Phys.* **30** 2854
Vladimirov V S and Volovich I V 1989 *Commun. Math. Phys.* **123** 659
[21] Zelenov E I 1991 *Theor. Math. Phys.* **86** 143
Zelenov E I 1993 *Commun. Math. Phys.* **155** 489
Zelenov E I 1994 *Commun. Math. Phys.* **159** 539

[22] Rammal R, Toulouse G and Virasoro M A 1986 *Rev. Mod. Phys.* **58** 765
Brekke L and Freund P 1993 *Phys. Rep.* **233** 1
Vladimirov V S, Volovich I V and Zelenov E I 1994 *p-Adic Analysis and Mathematical Physics* (Singapore: World Scientific)

[23] Cassels J W S and Frohlich A (ed) 1968 *Algebraic Number Theory* (London: Academic)
Ramakrishnan D and Valenza R J 1999 *Fourier Analysis on Number Fields* (Berlin: Springer)
Taibleson M H 1975 *Fourier Analysis on Local Fields* (Princeton, NJ: Princeton University Press)
Weil A 1973 *Basic Number Theory* (Berlin: Springer)

[24] Gel'fand I M and Graev M I 1963 *Russ. Math. Surveys* **18** 29
Gel'fand I M, Graev M I and Piatetskii-Shapiro I I 1990 *Representation Theory and Automorphic Functions* (London: Academic)

[25] Benedetto J J and Benedetto R L 2004 *J. Geom. Anal.* **14** 423
Benedetto R L 2004 *Contemporary Math.* **345** 27

[26] Flornes K, Grossmann A, Holschneider M and Torresani B 1994 *Appl. Comput. Harmon. Anal.* **1** 137
Khrennikov A Yu and Kozyrev S V 2005 *Appl. Comp. Harmonic Anal.* **19** 61
Lang W C 1996 *SIAM J. Math. Anal.* **27** 305
Lang W C 1998 *Int. J. Math. Math. Sci.* **21** 307
Lang W C 1998 *Houston J. Math.* **24** 533

[27] Dragovich B 1995 *Int. J. Mod. Phys.* A **10** 2349
Djordjevic G S, Nesic L J and Dragovich B 1999 *Mod. Phys. Lett.* A **14** 317
Djordjevic G S and Dragovich B 2000 *Theor. Math. Phys.* **124** 1059
Dragovich B 2003 arXiv:hep-th/0312046v1

[28] Ribes L and Zalesskii P 2000 *Profinite Groups* (Berlin: Springer)
Wilson J 1998 *Profinite groups* (Oxford: Clarendon)

[29] Zak J 1967 *Phys. Rev. Lett.* **19** 1385
Brezin J 1970 *Trans. Am. Math. Soc.* **150** 611
Janssen A J E M 1982 *J. Math. Phys.* **23** 720